Ext-Enhanced NilHecke

Sunday, January 30, 2022 11:18 AM

- Recall $NH_m \subseteq \mathrm{End}_{\mathbb{Z}}(\underbrace{\mathbb{Z}[x_1,\dots,x_m]}^{=:R_m})$ gen by $\{\partial_i\}$ and mult by elements of $R=R_m$
- Recall

$$NH_m \cong \mathrm{End}_{\mathrm{Sym}_m}(R_m) \cong M_{[m]!}(R_m^{S_m})$$

Def: $\hat{R}_m = \mathbb{Z}[x_1,\dots,x_m] \otimes \Lambda[w_1,\dots,w_m]$

Lem: $S_m \circlearrowright \hat{R}_m$ via

think of as fundamental weights

- $s_i(x_j) = x_{s_i(j)}$
- $s_i(w_j) = w_j + \delta_{ij}(x_i - x_{i+1})w_{i+1}$

extra

- $s_i(fg) = s_i(f)s_i(g)$

Def For $1 \le i \le n-1$, $f \in \hat{R}_m$,

$$\partial_i(f) := \frac{f - s_i(f)}{x_i - x_{i+1}}$$

Ex: $\partial_i(w_j) = -\delta_{ij} w_{i+1}$

Exer: $\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g) \quad (*)$

Rem1: If $f \in R^{S_i}$, $\partial_i(f) = 0$, and $\partial_i : R \to R^{S_i}$

Def $\widehat{NH}_m \subseteq \mathrm{End}_{\mathbb{Z}}(\hat{R}_m)$ gen by $\{\partial_i\}$ and mult by elements of $\hat{R}_m$

- $\widehat{NH}_m$ is bigraded (λ, q) $\lambda = $ coho, $q = $ int

$|x_i| = (0,2)$ $\quad |\partial_i| = (0,-2)$ $\quad |w_i| = (1, -2i)$ 2

Prop 2: $\widehat{NH}_m \cong \dfrac{NH_m \otimes \Lambda[\underline{w}_m]}{((1),(2) \text{ w/o } f, (3))}$ as alg

(1) $\partial_i(w_k f) = w_k \partial_i(f)$ $\quad k \neq i$ $\quad R^{S_i}$

(2) $\partial_i((w_i - x_{i+1}w_{i+1})f) = (w_i - x_{i+1}w_{i+1})\partial_i(f)$ $\quad R^{S_i}$

(3) $w_i x_j = x_j w_i$

Pf: NilHecke relations proved in NH_m via induction and $(*)$ $\Rightarrow$ nilHecke relations hold in $\widehat{NH}_m$ $\Rightarrow$ just need to see how ele of NH_m commute w/ $\Lambda[\underline{w}_m]$ which is $(1),(2),(3)$.

- By Rem 1, if $r \in \hat{R}_m^{S_m}$, $\partial_i(rx) = r\partial_i(x)$

$\Rightarrow \widehat{NH}_m \longrightarrow \mathrm{End}^s_{\hat{R}_m^{S_m}}(\hat{R}_m)$

$\| \quad \{T \in \mathrm{End}_{\mathbb{Z}}(\hat{R}_m) \mid T(rx) = (-1)^{|r||T|} r T(x) \ \forall r \in \hat{R}_m^{S_m}\}$

part of $|T|$

- We use super version b/c mult by $w_i \curvearrowright \hat{R}_m$ is supercommutative.

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2. Extended symmetric polynomials

Lem 3: $\hat{R}_m^{S_m} = \bigcap_{i=1}^{n-1} \ker \partial_i$

Pf: Rem 1 $\Rightarrow$

$$\bigcap_{i=1}^{n-1} \mathrm{im}\,\partial_i \subseteq \hat{R}_m^{S_m} \subseteq \bigcap_{i=1}^{n-1} \ker \partial_i$$

half the q-deg

Trick: $\partial_i^2 = 0 \rightsquigarrow (\hat{R}_m, \partial_i)$ is a chain complex

NilHecke relation: $\partial_i x_i - x_{i+1}\partial_i = 1$ gives homotopy

$\mathrm{id} \sim 0 \Rightarrow (\hat{R}_m, \partial_i)$ is contractible

$\Rightarrow H^k(\hat{R}_m, \partial_i) = 0 \Rightarrow \ker \partial_i = \mathrm{im}\,\partial_i$ $\square$

Exer 4: Let w_0 be longest element in S_m. Check

$$\partial_{w_0} : \hat{R}_m \longrightarrow \hat{R}_m^{S_m}$$

- Thus can produce elements of $\hat{R}_m^{S_m}$ easily

Def Let $k \le m$. A superpartition of type $(m|k)$ is a pair (α, β), α is a partition w/ at most m parts, β a strict partition at most k parts,

$$\beta = \{(\beta_1, \ldots, \beta_k) \in \mathbb{N}_0^k \mid 0 \le \beta_1 < \beta_2 < \ldots < \beta_k \le m\}$$

Ex: The strict partitions of 3 are (3), $(1,2)$

- Let P_k^{str} = strict partitions w/ k parts

Def Given a superpartition (α, β) of type $(m|k)$ define the extended Schur polynomial

$$s_{\alpha,\beta}(\underline{x}_m, \underline{w}_m) := \partial_{w_0}(\underline{x}_m^{\delta + \alpha} \underline{w}_\beta)$$

where $\underline{x}_m^{\delta+\alpha} = x_1^{m-1+\alpha_1} x_2^{m-2+\alpha_2} \cdots x_m^{\alpha_m}$, $\underline{w}_\beta = w_{\beta_1} \cdots w_{\beta_k}$

__Rem:__ For $\beta = 0$, recover usual Schur poly

- $s_{\alpha,\beta}(\underline{x}_m, \underline{w}_m) = s_{\alpha,0}(\underline{x}_m, \underline{w}_m) w_\beta + \sum_{\mu > \beta} c_\mu(\underline{x}_m) w_\mu$ ($\in R_m$; lex order)

__Prop 5:__ $(\hat{R}_m^{S_m})_{(k,\cdot)}$ has a $R_m^{S_m}$-basis by

$$\{ s_{(0,\beta)} \mid \beta \in P_k^{str} \}$$

__Pf:__ Let $z_v \in \hat{R}_m^{S_m}$. By Prop 2, write

$$z_v = b_v w_v + \sum_{\mu > v} b_\mu w_\mu, \quad b_\mu \in R_m$$

__Claim:__ $b_v \in R_m^{S_m}$

__Pf:__ $0 = \partial_i(z_v) = \partial_i(b_v) w_v + s_i(b_v) \partial_i(w_v) + \sum_{\mu > v} \partial_i(b_\mu w_\mu)$

- Note ∂_i increases lex order on P_k^{str} $\Rightarrow$

$\partial_i(b_v) w_v = 0 \Rightarrow \partial_i(b_v) = 0 \ \forall i \overset{L3}{\Rightarrow} b_v \in R_m^{S_m}$

$\Rightarrow b_v s_{0,v} \in \hat{R}_m^{S_m} \Rightarrow$

$$z_v - b_v s_{(0,v)} = \sum_{\mu > v} b'_\mu w_\mu \in \hat{R}_m^{S_m}$$

so result follows from induction.

__Ex:__ $\hat{R}_2^{S_2}$ is a free $R_2^{S_2}$ mod of rank 4 w/ basis

$$s_{(0),(0)} = 1, \quad s_{(0),(1)} = w_1 - x_2 w_2$$

$$s_{(0),(2)} = w_2, \quad s_{(0),(1,2)} = (w_1 - x_2 w_2) w_2$$

__Cor 6:__ There is iso of bigraded superalg

$$\hat{R}_m^{S_m} \cong R_m^{S_m} \otimes \Lambda^\bullet(s_{(0,1)}, \dots, s_{(0,m)})$$

- Recall usual Schubert polynomials

$$w \in S_m \rightsquigarrow b_w := \partial_{w^{-1} w_0}(x_1^{m-1} x_2^{m-2} \cdots x_{m-1})$$

__Prop 7:__ $\hat{R}_m$ is a free $\hat{R}_m^{S_m}$-mod of graded rank $[m]!$ w/ basis given by $\{b_w\}_{w \in S_m}$

Motivation

Cor 8: (i) We have isomorphisms of superalgs

$$\widehat{NH}_m \cong \mathrm{End}_{\hat{R}_m^{S_m}}(\hat{R}_m) \cong M_{[m]!}(\hat{R}_m^{S_m})$$

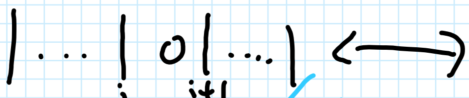
(ii) As a bigraded supermod over itself

$$\widehat{NH}_m \cong \hat{R}_m^{\oplus [m]!}$$

(iii) The supercenter of $\widehat{NH}_m$ is

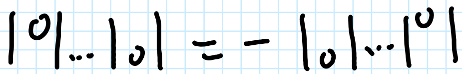
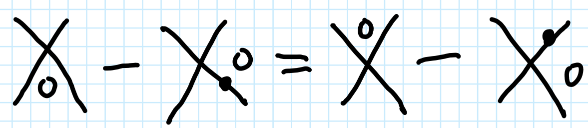
$$Z^s(\widehat{NH}_m) \cong \hat{R}_m^{S_m}$$

Diagrammatic presentation of $\widehat{NH}_m$

New gen:  $\longleftrightarrow w_i$

isotopy

New rels:

-  $\longleftrightarrow w_i w_j = -w_j w_i$
- 

2. Categorification of $M(q^\lambda)$

Motivation: By def

$$M(\lambda) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\lambda \overset{\text{set}}{\cong} U(\mathfrak{n}^-) \otimes_{\mathbb{C}} \mathbb{C}_\lambda$$

know how to categorify!

- Thus we just need to categorify action

Problem: $[E,F] = \dfrac{K-K^{-1}}{q-q^{-1}}$ how to categorify division??

Lauda's solution: Only allow rep w/ wt decomp so $[E,F]|_j = [j]|_j$ just grading shift

- $M(\lambda)$ is a wt module so this might work, but need different cat as λ varies
- Notice $M(\lambda)$ can be described via

$$\begin{tikzcd}
\cdots\ m_{i+1} \arrow[r, bend right, "{[\lambda,-i]}"'] & m_i \arrow[l, bend right, "{[i+1]}"'] \arrow[loop below, "\lambda q^{-2i}"] & \cdots\ m_2 \arrow[r, bend right, "{[\lambda,-1]}"'] & m_1 \arrow[l, bend right, "{[2]}"'] \arrow[r, bend right, "{[\lambda,0]}"'] & m_0 \arrow[l, bend right, "{[1]}"'] \arrow[loop below, "\lambda"]
\end{tikzcd}$$

—	$= F$
—	$= E$
—	$= K$

where $[\lambda, r] = \dfrac{\lambda q^r - \lambda^{-1} q^{-r}}{q - q^{-1}}$

- Thus, it might be more efficient to treat λ as a formal parameter and categorify universal Verma mod $M(\lambda)$ (instead of L)
- This allows for more leeway when categorifying relations $\longleftrightarrow$ finding isomorphisms b/t bimod

e.g, suppose $[M_i] = m_i$, $[\mathcal{E}] = E$

Show $\mathcal{E}(M_1) \cong \dfrac{\lambda}{q-q^{-1}} M_0 \oplus - \dfrac{\lambda^{-1}}{q-q^{-1}} M_0$ still have to make sense of this

v.s. $\mathcal{E}(M_1) \cong [n] M_0 \quad (\lambda = q^n)$

2.1 Unraveling $\frac{\lambda}{q-q^{-1}}$

Note $\dfrac{1}{q-q^{-1}} = \dfrac{-1}{q^{-1}(1-q^2)} = -q(1+q^2+q^4+\ldots)$

this makes sense now

- Know q corresponds to grading shift $\rightsquigarrow$ so λ should also be a grading shift
- $-$ signs can be categorified via supercat

Def A supercat $\mathcal{C}$ is a cat w/ functor $\Pi : \mathcal{C} \to \mathcal{C}$ s.t. $\Pi^2 \cong 1_{\mathcal{C}}$ + coherence axioms.
If $\mathcal{C}$ is additive, then $K_\oplus(\mathcal{C})$ is a mod over $\mathbb{Z}[\pi]/(\pi^2-1)$

- Specializing $\pi = -1$ gives $[\Pi(N)] = -[N]$

Ex: $\mathcal{C} = A\text{-smod}$, A superalgebra. If $N \in \mathcal{C}$
$N = N_0 \oplus N_1$, then $\Pi(N) = \Pi(N)_0 \oplus \Pi(N)_1$

$\Pi(N)_0 := N_1$, $\Pi(N)_1 := N_0$, $a \cdot n = (-1)^{|a|} a \cdot n$

Rmrk: $\{\text{Chain complexes}\}$ "=" $\{k[x]/x^2 - \text{smod}\}$

Rmrk: $\widehat{NH}_m = (\lambda\text{-deg even}) \oplus (\lambda\text{-deg odd})$

2.2 Categorification of $M(\lambda q^{-1})$

- Recall $\bigoplus_{m\geq 0} K_0^{\oplus}(NH_m\text{-gpmod}_f) \overset{alg}{\cong} U_q(n^-)$

$m \longleftrightarrow V\circ W = \mathrm{Ind}_{k,\ell}^{k+\ell}(V\boxtimes W)$

$\Delta \longleftrightarrow \mathrm{Res}(V) = \bigoplus_{a+b=k}\mathrm{Res}^{k}_{a,b}(V)$

- Now under inclusion $i_m: \widehat{NH}_m \hookrightarrow \widehat{NH}_{m+1}$

$\widehat{\mathrm{Ind}}_m^{m+1}: \widehat{NH}_m\text{-smod} \to \widehat{NH}_{m+1}\text{-smod}$

$\widehat{\mathrm{Res}}_m^{m+1}: \widehat{NH}_{m+1}\text{-smod} \to \widehat{NH}_m\text{-smod}$

- Define $Q = \Pi(-)\otimes_{\mathbb{Z}} q\,k[y]$ $\left(-q(1+q^2+\dots) = \frac{1}{q-q^{-1}}\right)$

$F_m: \widehat{\mathrm{Ind}}_m^{m+1}(-)$, $E_m = q^{2(m+1)}\lambda^{-1}\widehat{\mathrm{Res}}_m^{m+1}(-)$

Prop (Naisse-Vaz); $\exists$ SES (nonsplit) of functors

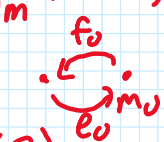
$0\to F_{m-1}E_{m-1}\to E_m F_m \to \lambda q^{1-2m}Q \oplus \Pi\lambda^{-1}q^{-(1-2m)}Q \to 0$

"Pf:" ① really a SES of bimodules

② use diagrammatic presentation of $\widehat{NH}_m$

Ex (Before): Let $f_m = \mathrm{Ind}_m^{m+1}: NH_m\text{-mod} \to \dots$

$e_m = \dots$, $NH_0 = \mathbb{Z}$, $NH_1 = \mathbb{Z}[x]$



$e_0 f_0(\mathbb{Z}) = q^2\mathbb{Z}[x] = q^2+\dots$ in $K_0(\mathbb{Z})_q$

Ex (After): $\widehat{NH}_0 = \mathbb{Z}$, $\widehat{NH}_1 = \mathbb{Z}[x]\otimes\Lambda[w]$

$E_0F_0(\mathbb{Z}) = q^2\lambda^{-1}(\underbrace{\mathbb{Z}[x]}_{1+q^2+\dots}\otimes\underbrace{\Lambda[w]}_{(1+\pi\lambda^2q^{-2})})$

$= \frac{q^2(\lambda^{-1}+\pi\lambda q^{-2})}{-q(q-q^{-1})} = \frac{\lambda q^{-1}-\lambda q}{q-q^{-1}}$ in $\tilde{K}_0(\mathbb{Z})^{\pi=-1}_{q,\lambda}$

Thrm (Naisse-Vaz) The functors $F = \bigoplus_{m\geq 0} F_m$, $E = \bigoplus_{m\geq 0} E_m$ give action of $U_q(sl_2)$ on $\tilde{K}_0(\widehat{NH})_{q,\lambda}$

s.t. (1) $\tilde{K}_0(\widehat{NH}_{lf})^{\pi=-1}_{q,\lambda} \overset{mod}{\cong} M(\lambda q^{-1})$

Categorification of L(q^n)

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(2) $\widetilde{K_0}(\widehat{NH}_m\text{-mod})_{q,\lambda}^{\tau=-1} \cong M(\lambda q^{-1})_{\lambda q^{-2m}}$

Pf: (2) B/c $\widehat{NH}_m \cong M_{[m]!}(\widehat{R}_m^{S_m})$

$\Rightarrow$ only 1 simple $\widehat{NH}_m$-mod up to gradings

$\Rightarrow$ LHS of (2) is 1-dim, and so is RHS

(1) Prop $\Rightarrow$ have $U_q(sl_2)$ action after taking $\widetilde{K_0}$

We just showed $[\widehat{NH}_0]$ has wt λq^{-1}, also h.w $\square$

3. Categorification of $L(q^n)$

- Recall $\lambda \longleftrightarrow$ "coho deg"
- It turns we can equip $\widehat{NH}_m$ w/ differentials d_n, $\forall n \in \mathbb{Z}^{\geq 0}$, s.t. $\widehat{NH}_m$ is a dg alg

• $d_n(w_i) = (-1)^{n-i} h_{n-i+1}(\underline{x}_i)$ = $h_{n-i+1}(x_1, \dots, x_i)$ complete homogeneous

• $d_n(x_j) = 0$ • $d_n(\partial_k) = 0$

- Notice that $d_n(w_1) = (-1)^{n-1} h_n(x_1)$
$= (-1)^{n-1} x_1^n$

- It turns out only need w_1 to gen $\widehat{NH}_m$ $\Rightarrow$ The $\lambda/2$-deg complex is of form

$$\cdots \longrightarrow NH_m \otimes w_1 \xrightarrow{d_n} \underline{NH_m}$$

$\Rightarrow H^0(\widehat{NH}_m, d_n) = NH_m^n$ ← cyclotomic quotient

Thrm (N-V): $H^*(\widehat{NH}_m, d_n)$ is concentrated in λ-deg 0 $\Rightarrow (\widehat{NH}_m, d_n)$ is formal

Cor: $D^c(\bigoplus_{m\geq 0} \widehat{NH}_m, d_n) \cong D^b(\bigoplus_{m\geq 0} NH_m^n, 0)$

Cor:

$K_0(D^c(\bigoplus_{m\geq 0} \widehat{NH}_m, d_n)) = K_0(\quad) = L(q^n)$